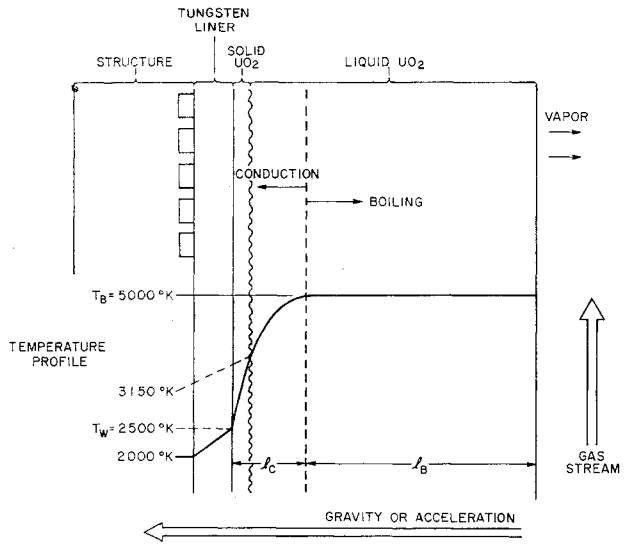

Fig. 1 Temperature profile through the liquid layer and container wall liner.

per second per unit volume, and κ is the thermal conductivity of the liquid. The thermal conductivity of UO_2 is 0.005 cal/sec/cm/°K at 2000°K. It will be assumed that it remains at this value up to 5000°K. The solution of this equation with appropriate boundary conditions yields

$$q(\text{wall}) = [2\kappa Q(T_B - T_W)]^{1/2} \quad (3)$$

and the thickness of the fuel region in which conduction to the wall occurs is

$$l_c = \{[2\kappa(T_B - T_W)]/Q\}^{1/2} \quad (4)$$

For the temperatures shown in Fig. 1 and for $Q = 1$ kw/cm^3,

$$q(\text{wall}) = 320 \text{ w/cm}^2 \quad (5)$$

$$l_c = 0.32 \text{ cm} \quad (6)$$

and if we choose 2000°K as the temperature of the outer surface of the liner, the liner thickness can be

$$l_w = 1.83 \text{ cm} \quad (7)$$

assuming it is tungsten with a thermal conductivity of 0.28 cal/cm °K sec. If the total thickness of liquid is 10 cm, the heat removed by boiling must be

$$q(\text{boiling}) = l_B Q \approx 10 \text{ kw/cm}^2 \quad (8)$$

On comparing q(wall) with q(boiling), it can be seen that the heat lost to the container is only a small fraction of the heat delivered by boiling, and its magnitude is such that it can easily be handled by conventional cooling techniques. This result assumes that within the liquid layer there is no convection, natural or otherwise. However, since the liquid is being cooled from below, natural convection should not occur.

For space applications, the reactor configuration might take the form of an externally moderated cavity with the liquid fuel held in a rotating drum similar to the configurations that have been proposed for dust and (bubbling) liquid core reactors (i.e., Refs. 2 and 3).

As an example, for the conditions previously cited, the delivery of 100 Mw of heat would require one square meter of liquid surface. This could be provided in a drum approximately one-half meter in diameter and one-half meter long. Since the heat load on the liner goes only as the square root of Q, it should be possible to use much higher rates of heat release if so desired.

In the preceding example, if we assume that all condensed uranium dioxide droplets are spun out of the gas before it leaves the reactor at 4000°K, then the UO_2 remaining in vapor form will constitute about 3 mole% of the exhaust gas stream. As a result of the difference in molecular weight, the weight flow rate of UO_2 will be approximately equal to the weight flow rate of helium. This is clearly unsuitable for a thermal rocket but could be tolerated in an MHD power cycle.

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Constraint Surface Normals for Structural Synthesis Techniques

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MANY modern techniques for the optimum synthesis of structures are based upon the concept of a design space. In an S-dimensional design space, each Cartesian coordinate axis represents a design variable, and thus a point in this space $\mathbf{D} = (d_1, d_2, \ldots, d_S)$ represents a design. Furthermore, each design has associated with it a value of the objective function (weight, cost, etc.) and has a behavior as determined by the application of an analysis method. In the typical synthesis problem the design corresponding to the minimum of the objective function is sought subject to inequality constraints on the behavior and perhaps on the design variables themselves.

In most realistic structural problems the functions defining the constraint surfaces cannot be expressed explicitly in terms of the design variables because the performance or behavior quantities (stress, displacement, etc.) themselves cannot be expressed explicitly in terms of the design variables. This fact has made the application of some of the more advanced nonlinear mathematical programing techniques[1–3] difficult if not impossible because a common feature of these highly directed methods is a need for the normal vectors to the locally active constraint surfaces. In some cases[4] these vectors have been obtained by the application of a finite difference scheme in which a "star" of nearby designs is analyzed. The setting up and solution of these "perturbed" designs is a time consuming and costly operation, and much structural synthesis work (e.g., Refs. 5 and 6) has sought to develop efficient methods of moving about in the design space without the benefit of this knowledge of the constraint surface orientation.

This note proposes a simple method for obtaining the normals to the constraint surfaces in structural (or other) synthesis problems where a linear analysis method is appropriate for predicting the behavior of the structure. Under certain common conditions the components of the normals so calculated will be exact; otherwise they will be approximate but are obtained with considerably less effort than by complete finite difference and are likely to be more accurate.

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Consider a problem the analysis of which is formulated as a set of matrix equations:

$$A\mathbf{X}_k = \mathbf{B}_k \qquad k = 1, 2, \ldots, L \tag{1}$$

where the $\mathbf{B}_k$ are L distinct load vectors, and $\mathbf{X}_k$ are L vectors of behavior variables such as stress and displacement, and A is a matrix that will be assumed to be assembled by a closed computer routine that has as inputs $\mathbf{D}$, i.e., $A = [a_{ij}(d_1, \ldots, d_S)]$.

A behavior constraint surface (as opposed to a side constraint on the d_s themselves) is said to be encountered in a synthesis process in design space when one of the two following possibilities occurs: 1) an element of some $\mathbf{X}_k$ is at its upper or lower limit, or 2) a function of the components of some $\mathbf{X}_k$ is at its upper or lower permitted value.

A common example of 2) is a limit on stress when Eq. (1) represents a displacement-type analysis. In this case

$$\boldsymbol{\sigma}_k = R\,\mathbf{X}_k \tag{2}$$

where $\boldsymbol{\sigma}_k$ is an m component vector of stresses for the kth load condition, $\mathbf{X}_k$ is an n component vector of displacement variables for the kth load condition, and $R = [r_{ij}]$ is an $m \times n$ matrix representing the coefficients of the stress-displacement relations. Often the elements of the R matrix do not depend upon $\mathbf{D}$.

A vector† normal to a constraint surface corresponding to $x_{ik} \leq$ (upper limit) or $x_{ik} \geq$ (lower limit) is

$$\nabla x_{ik} = (\partial x_{ik}/\partial d_1, \partial x_{ik}/\partial d_2, \ldots, \partial x_{ik}/\partial d_S) \tag{3}$$

evaluated at a point $\mathbf{D}$ where x_{ik} is equal to the limiting value. A vector normal to the constraint surface $\sigma_{pk} \leq$ (upper limit) or $\sigma_{pk} \geq$ (lower limit) is

$$\nabla \sigma_{pk} = \left(\sum_{j=1}^{n} r_{pj} \frac{\partial x_{jk}}{\partial d_1}, \ldots, \sum_{1=j}^{n} r_{pj} \frac{\partial x_{jk}}{\partial d_S} \right) \tag{4}$$

evaluated at a point where σ_{pk} is at the limiting value.

Thus it is seen that the special quantities needed are $\partial x_{ik}/\partial d_s$, $s = 1, 2, \ldots, S$. Differentiation of Eq. (1) with respect to d_s yields

$$A\,\partial \mathbf{X}_k/\partial d_s + [\partial A/\partial d_s]\mathbf{X}_k = 0 \tag{5}$$

assuming for simplicity that the loads are independent of the design. Since an analysis is necessary to establish that a point is on a constraint surface, both A^{-1} and the $\mathbf{X}_k$ are generally known at any point where the normals are sought. Premultiplying Eq. (5) by A^{-1} and rearranging

$$\partial \mathbf{X}_k/\partial d_s = -A^{-1}[\partial A/\partial d_s]\,\mathbf{X}_k \tag{6}$$

or in terms of components

$$\frac{\partial x_{ik}}{\partial d_s} = -\sum_{j=1}^{n} \sum_{p=1}^{n} a_{ij}^{(1)} \frac{\partial a_{jp}}{\partial d_s} x_{pk} \tag{6a}$$

where $a_{ij}^{(1)}$ denotes the ijth element of the inverse of A.

Since the matrix A is assembled by a computer program in many structural problems, the specific dependence of the a_{ij} upon the d_s is not known, and thus the $\partial a_{ij}/\partial d_s$ cannot ordinarily be calculated directly. One possibility is to calculate these quantities approximately by finite difference as

$$\partial a_{ij}/\partial d_s \simeq (\bar{a}_{ijs} - a_{ij})/\Delta d_s \tag{7}$$

where the notation

$$[\bar{a}_{ijs}] = \bar{A}_s = [a_{ij}(d_1, \ldots, d_s + \Delta d_s, \ldots, d_S)]$$

is introduced. This requires the assembly of a new matrix for the perturbed design but *not* its inversion; furthermore this rather crude difference formula will actually be exact when the a_{ij} are linearly dependent upon the design variables, i.e., when

$$a_{ij} = \alpha_{ij1}d_1 + \alpha_{ij2}d_2 + \ldots + \alpha_{ijS}d_S \tag{8}$$

In this case, which is a common one in structural problems,

$$\partial a_{ij}/\partial d_s \equiv (\bar{a}_{ijs} - a_{ij})/\Delta d_s \equiv \alpha_{ijs} \tag{9}$$

and $\partial a_{ij}/\partial d_s$ actually needs to be evaluated only once, since it does not depend upon $\mathbf{D}$. In fact, if the matrices $[\alpha]_s = [\alpha_{ijs}]$ are assembled and stored, the A matrix for any $\mathbf{D}$ can be assembled without the use of the original matrix assembly program as

$$A = \sum_{s=1}^{S} [\alpha]_s d_s$$

Note that, if the linear dependence [Eq. (8)] is known to hold, an efficient alternative to Eq. (7) is

$$\partial a_{ij}/\partial d_s \equiv a_{ij}(0, 0, \ldots, 0, 1, 0, \ldots, 0) \tag{10}$$

where the 1 appears in the sth position of $\mathbf{D}$.

A similar development is possible when the dependence of the elements of A is known to be of the form

$$a_{ij} = \sum_{s=1}^{S} \beta_{ijs} d_{s1} d_{s2} d_{s3} \ldots d_{sM} \tag{11}$$

where the d_{sp} are the design variables. This situation occurs, for example, in the stiffness matrix for a tubular truss structure where $M = 2$, and the variables d_{s1} and d_{s2} are the mean diameter D_s and the wall thickness T_s for the sth member.

An obvious disadvantage to disregarding the original matrix assembler is that the computer storage requirements for these special matrices may become excessive, in which event the required coefficients may be regenerated each time they are needed just as they must be when the form of the dependence is unknown or unmanageable. In any event, the implicit differentiation technique presented here is certainly more efficient and is likely to be more accurate than the direct finite difference method

$$\partial \mathbf{X}_k/\partial d_s \simeq 1/\Delta d_s\,[\mathbf{X}_{k(s)} - \mathbf{X}_k] \tag{12}$$

where $\mathbf{X}_{k(s)}$ denotes the solution to $\bar{A}_s\mathbf{X}_k = \mathbf{B}_k$ or

$$\partial \mathbf{X}_k/\partial d_s \simeq (1/\Delta d_s)[\bar{A}_s^{-1} - A^{-1}]\mathbf{B}_k \tag{13}$$

Even when the computational effort required by Eqs. (12) or (13) is reduced by efficient matrix solution methods, it will still be more extensive than that required by Eq. (6), since no additional inversions or solutions are required.

The question of how these constraint surface normals can best be utilized, or if they should be used at all, is still, of course, a subject of research. This is especially true when an implicit differentiation scheme similar to that described in this note is inapplicable (in the absence of a matrix formulation etc.).

† A *unit* vector normal to a surface $\phi =$ const is, of course, $\nabla\phi/|\nabla\phi|$.

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